Local gauge invariance implies Siegert's hypothesis

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The nonrelativistic Ward-Takahashi identity, a consequence of local gauge invariance in quantum mechanics, shows the necessity of exchange current contributions in case of nonlocal and/or isospin-dependent potentials. It also implies Siegert's hypothesis: in the nonrelativistic limit, two-body charge densities identically vanish. Neither current conservation, which follows from global gauge invariance, nor the constraints of (lowest order) relativity are sufficient to arrive at this result. Furthermore, a low-energy theorem for exchange contributions is established.

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In deriving the low-energy relation between nuclear electromagnetic current and charge multipoles, i.e. 'Siegert's theorem', Siegert conjectured that the charge density is in nonrelativistic order not modified by exchange contributions [1]. This is know as Siegert's *hypothesis* and it still is an essential ingredient in explicit calculations of nuclear electromagnetic processes (see, e.g., ref. [2] and references therein). In the original paper [1], using Fermi theory, corrections have been estimated to be of higher order in v/c, with v a typical velocity in the system. As will be demonstrated below, the lowest order relativity constraints [3] nevertheless do not *a priori* exclude a modification of the charge density. Furthermore, though one-boson exchange does not, it is claimed [4] that two-boson exchange does affect the charge density, thus conflicting Siegert's hypothesis and violating his theorem. In this Letter it is shown that the requirement of local gauge invariance yields Siegert's hypothesis. In nonrelativistic quantum mechanics, with moderate restrictions on the potentials, only the current density may be modified. This result, evidently in accordance with global gauge invariance, does not follow from current conservation.

The proof is based on the nonrelativistic Ward-Takahashi identity which was recently derived assuming local gauge invariance [5]. It relates the electromagnetic charge and current operators, $\rho, \vec{j}$, to the full propagator (Green's function), $G_E = (E - H)^{-1}$, of the strongly interacting N-particle system:

$$k_0 \rho(\vec{k}) - \vec{k} \cdot \vec{j}(\vec{k}) = G_{E'}^{-1} \rho_0(\vec{k}) - \rho_0(\vec{k}) G_E^{-1} \; . \tag{1}$$

Here $k_0 = E' - E$ is the energy, and $\vec{k}$ the momentum transfered by the (virtual) photon to the system. This relation is valid for off-shell energies E and E', i.e. not corresponding to the asymptotic energies of the initial and final states. Acting on those states, the inverse Green's functions just yield zero. Note that the right-hand side of the identity contains ρ_0, defined as the unmodified charge density, i.e. the sum of the one-body densities. It appears because it

generates the local gauge transformations for the particle degrees of freedom. In momentum space, it separately shifts the momenta of the initial and final charged particles by the photon momentum $\vec{k}$.

We will show now that the nonrelativistic Ward-Takahashi identity excludes lowest order exchange contributions to the charge density, i.e.,

$$\rho(\vec{k}) = \rho_0(\vec{k}), \tag{2}$$

up to order v/c. Whereas the current generally has two-body pieces, the nonrelativistic charge density is just the sum of the one-body charge densities. Let us start with the nonrelativistic Ward-Takahashi identity for the noninteracting N-particle system

$$k_0\rho_0(\vec{k}) - \vec{k}\cdot\vec{j}_0(\vec{k}) = (E'-T)\rho_0(\vec{k}) - \rho_0(\vec{k})(E-T)\ , \tag{3}$$

where T denotes the kinetic energy operator and $\vec{j}_0$ the unmodified current operator. Both operators are sums of the corresponding one-body operators. We include interactions via $H = H_0 + V = T + U + V$; U is defined as that part of the potential which commutes with the charge density, i.e. $[U, \rho_0] = 0$. As in ref. [5], time (energy) dependent interactions are excluded. One readily verifies that the unmodified charge and current operators also fulfil

$$k_0\rho_0(\vec{k}) - \vec{k}\cdot\vec{j}_0(\vec{k}) = (E'-H_0)\rho_0(\vec{k}) - \rho_0(\vec{k})(E-H_0)\ . \tag{4}$$

In case there are no other interactions ($V = 0$), local gauge invariance does not require two-body currents and eq. (1) is satisfied. If interactions are present which do not commute with ρ_0, for instance because of nonlocalities and/or isospin dependencies in V, exchange contributions are necessary. This immediately can be seen from the nonrelativistic Ward-Takahashi identity, rewritten by means of eq. (4),

$$(E'-E)\rho_e(\vec{k}) - \vec{k}\cdot\vec{j}_e(\vec{k}) = [\rho_0(\vec{k}), V]\ , \tag{5}$$

where $\rho_e = \rho - \rho_0\ ,\ \vec{j}_e = \vec{j} - \vec{j}_0$. Furthermore, since none of the appearing operators depends on energy and the relation is valid for arbitrary (off-shell) energies E and E', it immediately implies that $\rho_e = 0$ up to order v/c. Thus Siegert's hypothesis, cf. eq. (2), has been proven from the nonrelativistic Ward-Takahashi identity, eq. (1).

In this nonrelativistic framework with time-independent potentials the energy dependence in eq. (1) is rather trivial and, as a consequence, $E(E')$ only appears explicitly in eq. (5). This breaks down in a covariant approach, where analogous 'generalized Ward identities' have been derived by Kazes [6] a long time ago. However, due to the

energy-dependence of the operators involved, a similar conclusion about the two-body charge density cannot be drawn. Nevertheless, these relations can be exploited in, for example, deriving low-energy theorems (see, e.g., ref. [7]).

The off-energy shell implementation of electromagnetic current conservation,

$$\vec{k}\cdot\vec{j}(\vec{k}) = [H,\rho(\vec{k})]\ , \tag{6}$$

which recently appeared in the literature [8], reads

$$k_0\rho(\vec{k}) - \vec{k}\cdot\vec{j}(\vec{k}) = G^{-1}_{E'}\rho(\vec{k}) - \rho(\vec{k})G^{-1}_{E}\ . \tag{7}$$

The on-shell matrix element of the right-hand side of this relation also reduces to zero. Strictly speaking, local gauge invariance is not required here; a *global* symmetry already guarantees a conserved current. Since the electromagnetic field couples to a conserved current, eq. (7) should hold, irrespective of requiring a local $U(1)$ symmetry. In other words, eq. (1) is a stronger constraint than eq. (7). Simultaneously demanding both relations, i.e. the nonrelativistic Ward-Takahashi identity and current conservation, indeed indicates that the charge density is not modified by exchange contributions.

It is also instructive to consider the equation for the two-body current only imposing current conservation. Instead of eq. (5) one readily obtains

$$[H,\rho_e(\vec{k})] - \vec{k}\cdot\vec{j}_e(\vec{k}) = [\rho_0(\vec{k}),V]\ . \tag{8}$$

Eqs. (5) and (8) reduce to the same matrix element equation with respect to eigenstates of the Hamiltonian H. Obviously, as operator equations they are not equivalent. In particular, Siegert's hypothesis *only* follows from eq. (5), i.e. local gauge invariance. Indeed schemes for extracting exchange current contributions compatible with current conservation, cf. eqs. (6, 7, 8), however modifying the charge density have been developed [4]. Consequently, these currents violate the nonrelativistic Ward-Takahashi identity. Other extraction schemes, see e.g. [9], respect this identity, i.e. two-body charge densities vanish in the nonrelativistic limit.

It is important to realize that, because of the use of the electromagnetic four-potential A_μ, one cannot defer relativity completely even in noncovariant approaches like in ref. [5]. There it is of course tacitly assumed that the electromagnetic coupling is $A_\mu j^\mu$ and therefore $j^\mu(\vec{x}) = (\rho(\vec{x}),\vec{j}(\vec{x}))$ should transform as a four-vector [3,10]. In terms of the 'boost' operator $\vec{K}$ one has the conditions

$$[\vec{K},\rho(0)] = i\,\vec{j}(0)\,,$$

$$[K^m, j^n(0)] \; = i \, \delta^{nm} \rho(0). \tag{9}$$

The operator $\vec{K}$ can be expanded in v/c (or $1/m$, where m is the mass of the particles) [3,10]. For our purposes we only need the lowest order term,

$$\vec{K} \simeq \vec{K}_0 = M\vec{R}\,, \tag{10}$$

with the CM-coordinate $\vec{R}$ and the total mass M. Based on eq. (9), Friar gave a classification of exchange vector and axial currents [3]. Recall the usual realization in case of the vector current: ρ_e is of the same order as relativistic corrections to ρ_0, i.e. $(v/c)^2$. On the other hand, the exchange contribution $\vec{j}_e$ is of the same order as $\vec{j}_0$; both are $O(v/c)$, whereas relativistic corrections are of order $(v/c)^3$. This is actually compatible with the constraints above, cf. eq. (9), provided that

$$[K_0^m, j_e^n(0)] = 0\,. \tag{11}$$

However, the conditions (9) also allow for a *different* realization: if

$$[\vec{K}_0, \rho_e(0)] = 0\,, \tag{12}$$

then the exchange charge density, ρ_e, can be of nonrelativistic order, i.e. $O(1)$. In other words, based on the lowest order relativity constraints, one cannot *a priori* exclude two-body charge density contributions. Local $U(1)$ symmetry of the Schrödinger theory does exclude these.

An immediate consequence of this work is that it proves the low-energy theorem proposed in ref. [5]. Consider the two-particle system, $e_1 = e, e_2 = 0$, interacting via a nonlocal potential V. In lowest order of the photon momentum the exchange current is unambiguously given by

$$\vec{j}_e(\vec{k}) = e\frac{m_1}{m_1+m_2}\left[\nabla_{\vec{p}}\,\mathcal{V}(\vec{p}\,',\vec{p}) + \nabla_{\vec{p}\,'}\,\mathcal{V}(\vec{p}\,',\vec{p})\right], \tag{13}$$

where $\mathcal{V}$ denotes the matrixelement of V in terms of relative momenta after separating the CM-motion; m_1 and m_2 are the respective masses. Since the two-body charge density necessarily vanishes, this yields an unique prediction for $(\rho, \vec{j})$ in lowest order of $\vec{k}$. Note that the lowest order two-body current is purely longitudinal. Consequently, it does not contribute for real photons.

In this Letter we proved that Siegert's hypothesis follows from the requirement of local gauge invariance in nonrelativistic quantum mechanics. We believe that the gauge principle is rather universal, i.e. applicable to 'fundamental'

as well as 'effective' theories and, appropriately implemented, in classical as well as quantum mechanical formulations. Maxwell's equations, for instance, seem to apply well in the microscopic domain where the charge and current densities are treated quantum mechanically [11]. Given this observation, one indeed then can introduce the electromagnetic potentials, necessarily leading to local gauge invariance. In the quantum mechanical realization of this gauge symmetry, via the generator of the local phase transformations, the nonrelativistic Ward-Takahashi identity is valid. Nevertheless, an alternative view is possible. Local gauge invariance is supposed to be imposed in some underlying (more) fundamental theory, from which an effective model should be derived. The actual construction, however, usually not only involves approximations but also may be rather decoupled from the underlying theory. For instance, in introducing realistic nucleon-nucleon potentials a clear connection to QCD is lacking. One therefore may argue that imposing current conservation is sufficient and that local gauge invariance is too strong a condition. Clearly, this does not cover the point of view that constraints based on symmetries like $U(1)$ local invariance are useful in constructing effective models. These constraints, e.g. Ward-Takahashi identities, also provide consistency checks for practical calculations in such models. Our findings in this work, in particular the theoretical confirmation of the phenomenologically well-established Siegert hypothesis, support this latter point of view.

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